

# Symmetry of large solutions of nonlinear elliptic equations in a ball

Alessio Porretta \*

*Dipartimento di Matematica, Università di Roma Tor Vergata,  
Via della Ricerca Scientifica 1, 00133 Roma, Italia*

Laurent Véron

*Laboratoire de Mathématiques et Physique Théorique CNRS UMR 6083,  
Université François Rabelais, Tours 37200, France*

**Abstract** Let  $g$  be a locally Lipschitz continuous real valued function which satisfies the Keller-Osserman condition and is convex at infinity, then any large solution of  $-\Delta u + g(u) = 0$  in a ball is radially symmetric.

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## 1 Introduction

Let  $B_R$  denote the open ball of center 0 and radius  $R > 0$  in  $\mathbb{R}^N$ ,  $N \geq 2$ . A classical result due to Gidas, Ni and Nirenberg [9] asserts that, if  $g$  is a locally Lipschitz continuous real valued function, any  $u \in C^2(\overline{\Omega})$  which is a positive solution of

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases} \quad (1.1)$$

is radially symmetric. The proof of this result is based on the celebrated Alexandrov-Serrin moving plane method. Later on, this method was used in many occasions, with a lot of refinements for obtaining selected symmetry results and a priori estimates for solutions of semilinear elliptic equations. If the boundary condition is replaced by  $u = k \in \mathbb{R}$ , clearly the radial symmetry still holds if  $u - k$  does not change sign in  $B_R$ . Starting from this observation, it was conjectured by Brezis [5] that any solution  $u$  of

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } B_R \\ \lim_{|x| \rightarrow R} u(x) = \infty, \end{cases} \quad (1.2)$$

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is indeed radially symmetric. Notice that this problem admits a solution (usually called a “large solution”) if and only if  $g$  satisfies the Keller-Osserman condition:  $g \geq h$  on  $[a, \infty)$ , for some  $a > 0$  where  $h$  is non decreasing and satisfies

$$\begin{cases} \int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \\ \text{where } H(s) = \int_a^s h(t)dt. \end{cases} \quad (1.3)$$

Up to now, at least to our knowledge, only partial results were known concerning the radial symmetry of solutions of (1.2): in [14], the authors prove this result assuming (besides the Keller–Osserman condition) that  $g'(s)/\sqrt{G(s)} \rightarrow \infty$  as  $s \rightarrow \infty$ , or for the special case when  $g(s) = s^q$ , using the estimates for the second term of the asymptotic expansion of the solution near the boundary. Of course, the symmetry can also be obtained via uniqueness, however uniqueness is known under an assumption of *global* monotonicity and convexity ([12], [13]). Otherwise, it is easy to prove, by a one-dimensional topological argument, that uniqueness for problem (1.2) holds for almost all  $R > 0$  under the mere monotonicity assumption. However, if  $g$  is not monotone, uniqueness may not hold (see e.g. [1], [14], [16]), and it turns out to be very important to know whether *all* the solutions constructed in a ball are radially symmetric, a fact that would lead to a full classification of all possible solutions. Let us point out that the interest in such qualitative properties of large solutions has been raised in the last few years from different problems (see e.g. [1], [6], [7], [8] and the references therein).

In this article we prove that Brezis’ conjecture is verified under an assumption of asymptotic convexity upon  $g$ , namely we prove

**Theorem 1.1** *Let  $g$  be a locally Lipschitz continuous function. Assume that  $g$  is positive and convex on  $[a, \infty)$  for some  $a > 0$ , and satisfies the Keller-Osserman condition. Then any  $C^2$  solution of (1.2) is radially symmetric and increasing.*

Notice that the Keller-Osserman condition implies that the function  $g$  is superlinear at infinity. The convexity assumption on  $g$  is then very natural in such context.

In order to prove Theorem 1.1, we prove first a suitable adaptation of Gidas-Nirenberg moving-planes method to the framework of large solutions, without requiring any monotonicity assumption on  $g$ . This first result, which can have an interest in its own, reads as follows:

**Theorem 1.2** *Assume that  $g$  is locally Lipschitz continuous and let  $u$  be a solution of (1.2) which satisfies*

$$\begin{cases} \lim_{|x| \rightarrow R} \partial_r u = \infty \\ |\nabla_\tau u| = o(\partial_r u) \quad \text{as } |x| \rightarrow R, \quad \forall \tau \perp x \text{ s. t. } |\tau| = 1, \end{cases} \quad (1.4)$$

where  $\partial_r u$  and  $\nabla_\tau u$  are respectively the radial derivative and the tangential gradient of  $u$ . Then  $u$  is radially symmetric and  $\partial_r u > 0$  on  $B_R \setminus \{0\}$ .

Thus, in view of the previous statement, our main point in order to deduce the general result of Theorem 1.1 is to prove that condition (1.4) always holds (even in a stronger form) if we assume that  $g$  is asymptotically convex, and this is achieved by providing sharp informations on the radial and the tangential behavior of  $u$  near the boundary.

## 2 Proof of the results

Let  $\mathcal{B} = \{e_1, \dots, e_N\}$  be the canonical basis of  $\mathbb{R}^N$ . If  $P \in \mathbb{R}^N$  and  $\rho > 0$ , we denote by  $B_\rho(P)$  the open ball with center  $P$  and radius  $\rho$ , and for simplicity  $B_\rho(0) = B_\rho$ . We consider the problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } B_R \\ u(x) = \infty & \text{on } \partial B_R, \end{cases} \quad (2.1)$$

where  $R > 0$ . By a solution of (2.1), we mean that  $u \in C^2(B_R)$  is a classical solution in the interior of the ball and that  $u(x)$  tends to infinity uniformly as  $|x|$  tends to  $R$ .

We shall consider the following assumptions on  $g$ :

$$g : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is locally Lipschitz continuous.} \quad (2.2)$$

$$\exists a > 0 \text{ s.t. } g \text{ is positive and convex on } [a, \infty), \quad (2.3)$$

and satisfies

$$\int_a^{+\infty} \frac{1}{\sqrt{G(t)}} dt < +\infty, \quad \text{where } G(t) = \int_a^t g(s) ds. \quad (2.4)$$

Note that convexity and (2.4) imply that  $g$  is increasing on  $[b, \infty)$  for some  $b > 0$ .

If  $u \in C^1(B_R)$  we denote by  $\partial u / \partial r(x) = \langle Du(x), x / |x| \rangle$  the radial derivative of  $u$ , and by  $\nabla_\tau u(x) = (Du(x) - |x|^{-1} \partial u / \partial r(x))x$  the tangential gradient of  $u$ . Our first technical result, which is a reformulation in the framework of large solutions of the famous original proof of Gidas, Ni and Nirenberg [9], is the following

**Theorem 2.1** *Assume that  $g$  satisfies (2.2), and let  $u$  be a solution of (2.1). If there holds*

$$\begin{aligned} (i) \quad & \lim_{|x| \rightarrow R} \frac{\partial u}{\partial r}(x) = \infty \\ (ii) \quad & |\nabla_\tau u(x)| = o\left(\frac{\partial u}{\partial r}(x)\right) \quad \text{as } |x| \rightarrow R, \end{aligned} \quad (2.5)$$

*then  $u$  is radially symmetric and  $\partial u / \partial r > 0$  in  $B_R \setminus \{0\}$ .*

**Proof.** Since the equation is invariant by rotation, it is sufficient to prove that (2.5) implies that  $u$  is symmetric in the  $x_1$  direction.

We claim first that for any  $P \in \partial B^+ := \partial B_R \cap \{x \in \mathbb{R}^N : x_1 > 0\}$ , there exists  $\delta \in (0, R)$  such that

$$\frac{\partial u}{\partial x_1}(x) > 0 \quad \forall x \in B_R \cap B_\delta(P). \quad (2.6)$$

Indeed, thanks to (2.5) we have,

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial r} \frac{x_1}{|x|} + \left( Du - \frac{\partial u}{\partial r} \frac{x}{|x|} \right) \cdot e_1 \\ &= \frac{\partial u}{\partial r} \left( \frac{x_1}{|x|} + \left( \frac{\partial u}{\partial r} \right)^{-1} \left( Du - \frac{\partial u}{\partial r} \frac{x}{|x|} \right) \cdot e_1 \right) \\ &= \frac{\partial u}{\partial r} \left( \frac{x_1}{|x|} + o(1) \right) \quad \text{as } |x| \rightarrow R.\end{aligned}$$

Since  $P \in \partial B^+$ , the claim follows straightforwardly.

Next we follow the construction in [9]. For any  $\lambda < R$ , set  $T_\lambda$  the hyperplane  $\{x_1 = \lambda\}$  and  $\Sigma_\lambda = \{x \in B_R : \lambda < x_1 < R\}$ ,  $\Sigma'_\lambda = \{x \in B_R : 2\lambda - R < x_1 < \lambda\}$  the symmetric caps reflected with respect to  $T_\lambda$ ; denote also  $x_\lambda = (2\lambda - x_1, x_2, \dots, x_N)$  the reflected point and  $u_\lambda = u(x_\lambda)$  the reflected function, for  $x \in \Sigma_\lambda$ . Let  $P_0 = Re_1$  and let  $\delta = \delta(P_0) > 0$  be the real number such that (2.6) holds in  $B_R \cap B_{\delta_0}(P_0)$ . If  $\lambda_0 = R - \delta_0^2/2R$ , there holds

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } \Sigma_{\lambda_0} \cup \Sigma'_{\lambda_0},$$

so that, in particular,

$$u(x_\lambda) < u(x) \quad \text{and} \quad \frac{\partial u}{\partial x_1} > 0 \quad \text{in } \Sigma_\lambda, \quad (2.7)$$

for  $\lambda \geq \lambda_0$ . We define

$$\mu = \inf\{\lambda > 0 : \text{ s. t. (2.7) holds true } \}$$

and we claim that  $\mu = 0$ . We proceed by contradiction and assume that  $\mu > 0$ . Denote by  $K_\mu = T_\mu \cap \partial B_R$ : since  $K_\mu$  is compact, thanks to (2.6) there exists an  $\varepsilon$ -neighborhood  $U_\varepsilon$  of  $K_\mu$  such that

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } U_\varepsilon \cap B_R. \quad (2.8)$$

By definition of  $\mu$  there holds  $u \geq u_\mu$  in  $\Sigma_\mu$ ; thus, if we denote  $D_\varepsilon = B_{R-\varepsilon/2} \cap \Sigma_\mu$ , we have

$$\begin{cases} \Delta(u - u_\mu) = a(x)(u - u_\mu) & \text{in } D_\varepsilon \\ u - u_\mu \geq 0 & \text{in } D_\varepsilon, \end{cases} \quad (2.9)$$

where  $a(x) = (g(u) - g(u_\mu))/(u - u_\mu)$ . Thanks to (2.2) and since  $\overline{D_\varepsilon}$  is in the interior of  $B_R$ ,  $a(x)$  is a bounded function in  $D_\varepsilon$ , and the strong maximum principle applies to (2.9). Since  $u$  tends to infinity at the boundary and is finite in the interior, for  $\varepsilon$  small we clearly have  $u \not\equiv u_\mu$  in  $D_\varepsilon$ : therefore we conclude that  $u > u_\mu$  in  $D_\varepsilon$ , and, since  $u = u_\mu$  on  $T_\mu \cap \partial D_\varepsilon$  and  $\partial u_\mu / \partial x_1 = -\partial u / \partial x_1$  on  $T_\mu$ , it follows from Hopf boundary lemma that

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{on } T_\mu \cap \partial D_\varepsilon.$$

Since  $u \in C^1(B_R)$ , the last assertion, together with (2.8), implies that there exists  $\sigma > 0$  such that

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } B_R \cap \{x : \mu - \sigma < x_1 < \mu + \sigma\}. \quad (2.10)$$

Moreover, since  $\varepsilon$  can be chosen arbitrarily small, we deduce that

$$u > u_\mu \quad \text{in } \Sigma_\mu.$$

Now, by definition of  $\mu$ , there exists an increasing sequence  $\lambda_n$  converging to  $\mu$  and points  $x_n \in \Sigma_{\lambda_n}$  such that

$$u(x_n) \leq u((x_n)_{\lambda_n}). \quad (2.11)$$

Up to subsequences,  $\{x_n\}$  will converge to a point  $\bar{x} \in \bar{\Sigma}_\mu$ . However,  $\bar{x}$  cannot belong to  $\Sigma_\mu$ , since in the limit we would have  $u(\bar{x}) \leq u(\bar{x}_\mu)$  while we proved that  $u > u_\mu$  in  $\Sigma_\mu$ . On the other hand, we can also exclude that  $\bar{x} \in T_\mu$ ; indeed, we have

$$u(x_n) - u((x_n)_{\lambda_n}) = 2(x_n - \lambda_n) \frac{\partial u}{\partial x_1}(\xi_n)$$

for a point  $\xi_n \in ((x_n)_{\lambda_n}, x_n)$ . If (a subsequence of)  $x_n$  converges to a point in  $T_\mu$ , then for  $n$  large we have  $\text{dist}(\xi_n, T_\mu) < \sigma$  and from (2.10) we get  $u(x_n) - u((x_n)_{\lambda_n}) > 0$  contradicting (2.11). We are left with the possibility that  $\bar{x} \in \partial\Sigma_\mu \setminus T_\mu$ : but this is also a contradiction since  $u$  blows up at the boundary and it is locally bounded in the interior, so that  $u(x_n) - u((x_n)_{\lambda_n})$  would converge to infinity.

Thus  $\mu = 0$  and (2.7) holds in the whole  $\{x \in B_R : x_1 > 0\}$ . We deduce that  $u$  is symmetric in the  $x_1$  direction and  $\partial u / \partial x_1 > 0$ . Applying to any other direction we conclude that  $u$  is radial and  $\partial u / \partial r > 0$ .  $\blacksquare$

**Remark 2.1** Let us recall that in some special examples (for instance when  $g(s)$  has an exponential or a power-like growth) the asymptotic behavior at the boundary of the gradient of the large solutions has already been studied (see e.g. [2], [4], [17]) so that the previous result could be directly applied to prove symmetry. In general, through a blow-up argument, we are able to prove (2.5) if

$$s \mapsto \frac{g(s)}{\sqrt{G(s)}} \int_s^\infty \frac{1}{\sqrt{2G(\xi)}} d\xi \quad (2.12)$$

is bounded at infinity; however this assumption does not include the case when  $g$  has a slow growth at infinity (such as  $g(r) \equiv r(\ln r)^\alpha$  with  $\alpha > 2$ ) and is not so general as (2.3).

**Theorem 2.2** *Assume that (2.2), (2.3) and (2.4) hold. Then any solution  $u$  of (2.1) is radial and  $\partial u / \partial r > 0$  in  $B_R \setminus \{0\}$ .*

The following preliminary result is a consequence of more general results in [12], [13]. However we provide here a simple self-contained proof for the radial case.

**Lemma 2.1** *Let  $h$  be a convex increasing function satisfying the Keller-Osserman condition*

$$\int_a^{+\infty} \frac{ds}{\sqrt{H(s)}} < \infty, \quad H(s) = \int_a^s h(t)dt, \quad (2.13)$$

for some  $a > 0$ . Then the problem

$$\begin{cases} -\Delta v + h(v) = 0 & \text{in } B_R \\ \lim_{|x| \rightarrow R} v(x) = \infty, \end{cases} \quad (2.14)$$

has a unique solution.

**Proof.** Since  $h$  is increasing, there exist a maximal and a minimal solution  $\bar{v}$  and  $\underline{v}$ , which are both radial, so that it is enough to prove that  $\bar{v} = \underline{v}$ . To this purpose, observe that if  $v$  is radial we have  $(v'r^{N-1})' = r^{N-1}h(v)$  so that, since  $v'(0) = 0$ , and replacing  $H$  by  $\tilde{H} = H - H(\min v)$  which is nonnegative on the range of values of  $v$ , we have

$$\frac{(v'r^{N-1})^2}{2} = \int_0^r s^{2(N-1)} h(v) v' ds \leq r^{2(N-1)} \tilde{H}(v)$$

which yields

$$0 \leq v' < \sqrt{2\tilde{H}(v)}. \quad (2.15)$$

Define now  $w = F(v) = \int_v^\infty \frac{ds}{\sqrt{2\tilde{H}(s)}}$ . A straightforward computation, and condition (2.13), show that  $w$  solves the problem

$$\begin{cases} \Delta w = b(w) (|Dw|^2 - 1) & \text{in } B_R, \\ w = 0 & \text{on } \partial B_R, \end{cases} \quad (2.16)$$

where  $b(w) = h(v)/\sqrt{2\tilde{H}(v)}$ . One can easily check that the convexity of  $h$  implies that  $h(v)/\sqrt{2\tilde{H}(v)}$  is nondecreasing, hence  $b(w)$  is nonincreasing with respect to  $w$ . Moreover, since  $|Dw| = |w'| = v'/\sqrt{2\tilde{H}(v)}$ , from (2.15) one gets  $|w'| < 1$ . Note that the transformation  $v \mapsto w$  establishes a one-to-one monotone correspondence between the large solutions of (2.14) and the solutions of (2.16), so that  $\bar{w} = F(\underline{v})$  and  $\underline{w} = F(\bar{v})$  are respectively the minimal and the maximal solutions of (2.16). Thus we have

$$((\bar{w} - \underline{w})' r^{N-1})' = r^{N-1} [b(\bar{w})(|\bar{w}'|^2 - 1) - b(\underline{w})(|\underline{w}'|^2 - 1)] \geq r^{N-1} b(\bar{w})(|\bar{w}'|^2 - |\underline{w}'|^2),$$

so that the function  $z = (\bar{w} - \underline{w})' r^{N-1}$  satisfies

$$z' \geq a(r)z, \quad z(0) = 0, \quad \text{where } a(r) = b(\bar{w})(\bar{w}' + \underline{w}').$$

Because  $a$  is locally bounded on  $[0, R)$ , we deduce that  $z \geq 0$ , hence  $\bar{w} - \underline{w}$  is nondecreasing. Since  $\bar{w} - \underline{w}$  is nonnegative and  $\bar{w}(R) = \underline{w}(R) = 0$  we deduce that  $\bar{w} = \underline{w}$ , hence  $\bar{v} = \underline{v}$ . ■

**Lemma 2.2** *Assume that  $g$  satisfies (2.3) and (2.4), and that  $u$  is a solution of (2.1). Then*

$$\begin{aligned} (i) \quad & \lim_{|x| \rightarrow R} \nabla_\tau u(x) = 0 \\ (ii) \quad & \lim_{|x| \rightarrow R} \frac{\partial u}{\partial r}(x) = \infty, \end{aligned} \tag{2.17}$$

and the two limits hold uniformly with respect to  $\{x : |x| = r\}$ .

**Proof.** In spherical coordinates  $(r, \sigma) \in (0, \infty) \times S^{N-1}$  the Laplace operator takes the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_s u,$$

where  $\Delta_s$  is the Laplace Beltrami operator on  $S^{N-1}$ . If  $\{\gamma_j\}_{j=1}^{N-1}$  is a system of  $N-1$  geodesics on  $S^{N-1}$  crossing orthogonally at  $\tilde{\sigma}$ , there holds

$$\Delta_s u(r, \tilde{\sigma}) = \sum_{j \geq 1} \frac{d^2 u(r, \gamma_j(t))}{dt^2} \Big|_{t=0}. \tag{2.18}$$

On the sphere the geodesics are large circles. The system of geodesics can be obtained by considering a set of skew symmetric matrices  $\{A_j\}_{j=1}^{N-1}$  such that  $\langle A_j \tilde{\sigma}, A_k \tilde{\sigma} \rangle = \delta_j^k$ , and by putting  $\gamma_j(t) = e^{tA_j} \tilde{\sigma}$ .

*Step 1: two-side estimate on the tangential first derivatives.* By assumption (2.3)  $g$  can be written as

$$g(s) = g_\infty(s) + \tilde{g}(s),$$

where  $g_\infty(s)$  is a convex increasing function satisfying (2.4) and  $\tilde{g}(s)$  is a locally Lipschitz function such that  $\tilde{g} \equiv 0$  in  $[M, \infty)$  for some  $M > 0$ . In particular,  $u$  satisfies

$$\Delta u - g_\infty(u) = \tilde{g}(u).$$

Since  $u$  blows up uniformly, there holds  $u(x) \geq M$  if  $|x| \in [r_0, R)$  for a certain  $r_0 < R$ , hence

$$\tilde{g}(u) = \tilde{g}(u) \chi_{\{|x| \leq r_0\}} \leq K_0,$$

so that

$$|\Delta u - g_\infty(u)| \leq K_0. \tag{2.19}$$

Set  $\varphi(r) = \frac{1}{2N}(R^2 - r^2)$ , thus  $\varphi$  satisfies  $-\Delta \varphi = 1$  and  $\varphi = 0$  on  $\partial B_R$ . We deduce from (2.19) that

$$\Delta(u - K_0 \varphi) \geq g_\infty(u) \geq g_\infty(u - K_0 \varphi),$$

since  $g_\infty$  is increasing and  $\varphi$  is nonnegative. Thus  $u - K_0 \varphi$  is a sub-solution of the problem

$$\begin{cases} -\Delta v + g_\infty(v) = 0 & \text{in } B_R \\ \lim_{|x| \rightarrow R} v(x) = \infty. \end{cases} \tag{2.20}$$

Similarly  $u + K_0\varphi$  is a super-solution of the same problem. By Lemma 2.1, problem (2.20) has a unique solution  $U_R$ . By approximating  $U_R$  by the large solution  $U_{R'}$  of the same equation in  $B_{R'}$  with  $R' < R$  and  $R' > R$  we derive

$$U_R - K_0\varphi \leq u \leq U_R + K_0\varphi. \quad (2.21)$$

Since the problem (2.1) is invariant by rotation, for any  $j = 1, \dots, N-1$ , and any  $h \in \mathbb{R}$ , the function  $u^h$  defined by  $u^h(x) = u(e^{hA_j}(x)) = u(r, e^{hA_j}\sigma)$  is a solution of (2.1) and still  $\tilde{g}(u^h) = 0$  if  $r \in [r_0, R)$ , so that

$$\Delta u^h = g_\infty(u^h) \quad \text{if } r \in (r_0, R).$$

Since  $u \in C^1(B_R)$ , there holds

$$|u^h - u| \leq L|h| \quad \text{if } r = r_0.$$

Let us set

$$P(r) = \begin{cases} \frac{r^{2-N} - R^{2-N}}{r_0^{2-N} - R^{2-N}} & \text{if } N > 2 \\ \frac{\ln r - \ln R}{\ln r_0 - \ln R} & \text{if } N = 2, \end{cases} \quad (2.22)$$

and  $v^h(x) = u^h(x) + |h|LP(|x|)$ ; then  $\Delta v^h = \Delta u^h$ , and since  $g_\infty$  is increasing,

$$\Delta(v^h - u) \leq g_\infty(v^h) - g_\infty(u) \quad \text{in } B_R \setminus B_{r_0}. \quad (2.23)$$

Observe that  $u^h$ , as  $u$ , also satisfies (2.21), so that in particular

$$u^h(x) - u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow R. \quad (2.24)$$

Therefore  $v^h(x) - u(x) \rightarrow 0$  as  $|x| \rightarrow R$  too, while by construction  $v^h \geq u$  on  $\partial B_{r_0}$ . We conclude from (2.23) (e.g. using the test function  $(v^h - u + \varepsilon)_-$ , which is compactly supported, and then letting  $\varepsilon$  go to zero) that

$$v^h = u^h + |h|LP(r) \geq u.$$

We recall that the Lie derivative  $L_{A_j}u$  of  $u(r, \cdot)$  following the vector field tangent to  $S^{N-1}$   $\eta \mapsto A_j\eta$  is defined by

$$L_{A_j}u(r, \sigma) = \frac{du(r, e^{tA_j}\sigma)}{dt} \Big|_{t=0},$$

so we get, by letting  $h \rightarrow 0$ ,

$$|L_{A_j}u(r, \tilde{\sigma})| \leq LP(r) < C(R - r). \quad (2.25)$$



*Step 2: one-side estimate on the tangential second derivatives.* Next we define the function  $w^h$  by

$$w^h = \frac{u^h + u^{-h} - 2u}{h^2}.$$

As before, let  $r_0 < R$  be such that  $u \geq M$  on  $B_R \setminus B_{r_0}$ . Thus  $g(u) = g_\infty(u)$  on  $B_R \setminus B_{r_0}$ , and

$$\Delta w^h = \frac{1}{h^2} \left( g_\infty(u^h) + g_\infty(u^{-h}) - 2g_\infty(u) \right) \quad \text{on } B_R \setminus B_{r_0}.$$

Since  $g_\infty$  is convex, there holds

$$g_\infty(a) + g_\infty(b) - 2g_\infty(c) \geq \xi(a + b - 2c) \quad \forall \xi \in \partial g_\infty(c)$$

where  $\partial g_\infty(c) = [g'_\infty(c_-), g'_\infty(c_+)]$ . Hence

$$\Delta w^h \geq \xi_u w^h \quad \text{on } B_R \setminus B_{r_0},$$

for any  $\xi_u \in \partial g_\infty(u)$ . Since  $g_\infty$  is increasing, we have  $\xi_u \geq 0$ , therefore  $(w^h)_+$  is a subharmonic function in  $B_R \setminus B_{r_0}$ . As  $u \in C^2(B_R)$ , there exists  $\tilde{L} > 0$  such that

$$w^h \leq \tilde{L} \quad \text{on } \partial B_{r_0}.$$

Moreover from (2.25) we get that  $w^h = 0$  on  $\partial B_R$ . We conclude that

$$(w^h)_+(x) \leq \tilde{L} P(|x|),$$

where  $P(r)$  is defined in (2.22). Letting  $h$  tend to zero we obtain

$$\frac{d^2 u(r, e^{tA_j} \tilde{\sigma})}{dt^2} \Big|_{t=0} \leq \tilde{L} P(r) \quad \text{for } r \in [r_0, R]. \quad (2.26)$$

Using (2.18), and the fact that  $\tilde{\sigma}$  is arbitrary, we derive

$$\Delta_s u(r, \sigma) \leq (N-1) \tilde{L} P(r) \quad \forall (r, \sigma) \in [r_0, R) \times S^{N-1}. \quad (2.27)$$

*Step 3: estimate on the radial derivative.* Using (2.22) and (2.27) we deduce that

$$(\Delta_s u)^+(x) = o(1) \quad \text{uniformly as } |x| \rightarrow R.$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u}{\partial r} \right) &= r^{N-1} \left[ g(u) - \frac{1}{r^2} \Delta_s u \right] \\ &\geq r^{N-1} g_\infty(u) - o(1) \quad \text{uniformly as } r \rightarrow R. \end{aligned} \quad (2.28)$$

Now one can easily conclude: let  $z(r)$  denote the minimal (hence radial) solution of

$$\begin{cases} \Delta z = g_\infty(z) & \text{in } B_R \setminus B_{r_0} \\ z = \min_{\partial B_{r_0}} u & \text{on } \partial B_{r_0} \\ \lim_{r \rightarrow R} z = \infty. \end{cases}$$

We have  $u(x) \geq z(x)$  if  $|x| \in [r_0, R)$ , hence  $g_\infty(u) \geq g_\infty(z)$ . Because this last function is not integrable near  $\partial B_R$ , one obtains

$$\lim_{r \rightarrow R} \int_{r_0}^r s^{N-1} g_\infty(u(s, \sigma)) ds \rightarrow \infty \quad \text{uniformly for } \sigma \in S^{N-1}.$$

Clearly (2.28) implies

$$\frac{\partial u}{\partial r}(r, \sigma) \xrightarrow{r \rightarrow R} \infty \quad \text{uniformly for } \sigma \in S^{N-1}.$$

This completes the proof of (2.17). ■

**Proof of Theorem 2.2.** By assumptions (2.3) and (2.4), and Lemma 2.2, we deduce that  $u$  satisfies (2.5), hence we apply Lemma 2.1 to conclude. ■

Finally, let us point out that thanks to Lemma 2.2 and using the moving plane method as in Theorem 2.1, we can derive a result describing the boundary behaviour of any solution of

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Gamma_{R,r} = \{x \in \mathbb{R}^N : r < |x| < R\} \\ \lim_{|x| \rightarrow R} u(x) = \infty, \end{cases} \quad (2.29)$$

which extends a similar result in [9].

**Corollary 2.1** *Assume that  $g$  satisfies (2.2), (2.3) and (2.4). Then any solution of (2.29) satisfies (2.17) and verifies  $\partial_r u > 0$  on  $\Gamma_{R,(R+r)/2}$ .*

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